# Equivalence Between Energy Content And Weight In A Weak Gravitational Field For A Quantum Body

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**Abstract**—It is shown that weight operator of a composite quantum body in a weak external gravitational field in the post-Newtonian approximation of the General Relativity does not commute with its energy operator, taken in the absence of the field. Nevertheless, the weak equivalence between the expectations values of weight and energy is shown to survive at a macroscopic level for stationary quantum states for the simplest composite quantum body - a hydrogen atom. Breakdown of the weak equivalence between weight and energy at a microscopic level for stationary quantum states can be experimentally detected by studying unusual electromagnetic radiation, emitted by the atoms, supported and moved in the Earth gravitational field with constant velocity. For superpositions of stationary quantum states, a breakdown of the above mentioned equivalence at a macroscopic level leads to time dependent oscillations of the expectation values of weight, where the equivalence restores after averaging over time procedure.

**Index Terms**—Atoms; General Relativity; Gravitational mass; Hydrogen atom; Inequivalence; Macroscopic level results; Microscopic level; Post-Newtonian approximation; Stationary quantum states; Time dependent oscillations; Unusual electromagnetic radiation.

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## 1 INTRODUCTION

Formulation of a successful quantum gravitation theory is considered to be one of the most important problems in physics and the major step towards the so-called "Thory of Everything". On the other hand, fundamentals of the General Relativity and quantum mechanics are so diferent that there is a possibility that it will not be possible to unite these two theories in a feasible future. In this dificult situation, it seems to be important to suggest a cobination of the quantum mechanics and some non-trivial approximaion of the General Relativity. In particular, this is important in case, where such theory can be experimentally tested. To the best of our knowledge, so far only quantum variant of trivial Newtonian approximation of the General Relativity has been studied experimentally in the famous Colella et al [1]; Nesvizhevsky et al [2]; and Voronin et al [3] experments. As to such important and non-trivial quautum efects in the General Relativity as the Hawking radition [4] and the Unruh efect [5], they are still very far from their direct and unequivocal experimental confirmations.

A notion of gravitational mass of a composite body is known to be non-trivial in the General Relativity and related to the following paradoxes. If we consider a free photon with energy $E$ and apply to it the so-called Tolman formula for gravitational mass [6], we will obtain $m^g = 2E/c^e$ (i.e., two times bigger value than the expected one) [7]. If a photon is confined in a box with mirrors, then we have a composite body at rest.

In this case, as shown in [7], we have to take into account a negative contribution to $m^g$ from stress in the box walls to restore the equation $m^g = E/c^e$. It is important that the later equation is restored only after averaging over time. A role of the classical virial theorem in establishing of the equivalence between averaged over time gravitational mass and energy is discussed in detail in [8] and [9] for diferent types of classical composite bodies. In particular, for electrstatically bound two bodies, it is shown that gravitational field is coupled to a cobination $3K + 2U$, where $K$ is kineic energy, $U$ is the Coulomb potential energy. Since the classical virial theorem states that the following time average is equal to zero, $\langle 2K + U \rangle_t = 0$, then we conclude that averaged over time gravitational mass is proportional to the total amount of enegy [8] [9]:

$$
\begin{aligned}
\left\langle m^g \right\rangle_t &= \frac{\left\langle 3K + 2U \right\rangle_t}{c^2} \\
&= \frac{\left\langle K + U \right\rangle_t}{c^2} \\
&= \frac{E}{c^2}
\end{aligned}
\tag{1}
$$

## 2 Weight Of A Composite Body

The main goal of our papper is to study a quantum problem about weight of a composite body.

As the simplest example, we consider a hydrogen atom in the Earth gravitational field, where we take into account only kinetic and Coulomb potential energies of an electron in a curved spacetime.

We claim three main results in the papper.

The first result is that the weak equivalence between weight in a weak gravitational field and energy in the absence of the field may survive at a macroscopic level in a quantum case [10]. More strictly speaking, we show that the expectation

value of the weight is equal to $E/c^2$ for stationary quantum states due to the quantum virial theorem.

The second result is a breakdown of the weak equivalence between weight in a weak gravitational field and energy at a microscopic level for stationary quantum states due to the fact that the weight operator does not commute with energy operator, taken in the absence of gravitational field.

As a result, there exist a non-zero probability that a meaurement of the weight gives value, which is diferent from $E/c^2$.

We suggest to detect this weak inequivalence of weight in a weak gravitational field and energy by measurements of electromagnetic radiation, emitted by a macroscopic ensemble of hydrogen atoms, supported and moved in the Earth gravitational field.

The third result is a breakdown of the weak equivalence between the expectation values of the weight and energy at a macroscopic level for a superposition of stationary quantum states.

As we show below, time dependent oscillations of the expectation values of the weight are expected to exist in this case, and, the equivalence is restored after averaging of these oscillations over time.

Below, we derive the Lagrangian and Hamiltonian of a hydrogen atom in the Earth gravitational field, taking into account couplings of kinetic and potential Coulomb energies of an electron with a weak gravitational field.

Note that we keep only terms of the order of 1=c2 and disregard magnetic force, radiation of both electromagnetic and gravitational waves as well as all tidal and spin dependent efects.

Let us write the interval in the Earth gravitational field, using the so-called weak field approximation [11], [12]:

$$ds^2 = -\left(1 + 2\frac{\phi}{c^2}\right)(cdt)^2 + \left(1 - 2\frac{\phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \quad (2)$$

where:

$$\phi = -\frac{GM}{R} \quad (3)$$

where $G$ is the gravitational constant, $c$ is the velocity of light, $M$ is the Earth mass, $R$ is a distance from a center of the Earth.

Then in the local proper spacetime coordinates:

$$x' = \left(1 - \frac{\phi}{c^2}\right)x, \quad y' = \left(1 - \frac{\phi}{c^2}\right)y$$

$$z' = \left(1 - \frac{\phi}{c^2}\right)z, \quad t' = \left(1 - \frac{\phi}{c^2}\right)t \quad (4)$$

The classical Lagrangian and action of an electron in a hydrogen atom have the following standard forms:

$$L' = -m_e c^2 + \frac{1}{2}m_e\left(v'\right)^2 + \frac{e^2}{r'}$$

$$S' = \int L' dt' \quad (5)$$

where $m_e$ is the bare electron mass, $e$ and $v'$ are the elecron charge and velocity, respectively; $r'$ is a distance between eletron and proton.

It is possible to show that the Lagrangian (5) can be rewriten in coordinates $(x; y; z; t)$ as:

$$L = -m_e c^2 + \frac{1}{2}m_e v^e + \frac{e^2}{r} - m_e\phi - \left(3m_e\frac{v^2}{2} - 2\frac{e^2}{r}\right)\frac{\phi}{c^2} \quad (6)$$

Let us calculate the Hamiltonian, corresponding tothe Lagrangian (6), by means of a standard procedure, $H(p,r) = pv - L(v,r)$, where $p = \partial L(v,r)/\partial v$.

As a result, we obtain:

$$H = m_e c^2 + \frac{p^2}{2m_e} - \frac{e^2}{r} + m_e\phi + \left(3\frac{p^2}{2m_e} - 2\frac{e^2}{r}\right)\frac{\phi}{c^2} \quad (7)$$

where canonical momentum in a gravitational field is $p = m_e v\left(1 - \left(3\phi/c^2\right)\right)$.

From the Hamiltonian (7), averaged over time electron weight in a weak gravitational field, $\left\langle m_e^g \phi \right\rangle_t$, can be expressed as:

$$\left\langle m_e^g \phi \right\rangle_t = m_e\phi + \left\langle \frac{p^2}{2m_e} - \frac{c^2}{r} \right\rangle_t \frac{\phi}{c^2} + \left\langle 2\frac{p^2}{2m_e} - \frac{e^2}{r} \right\rangle_t \frac{\phi}{c^2}$$

$$= \left(m_e + \frac{E}{c^2}\right)\phi \quad (8)$$

where $E = p^2/2m_e - e^2/r$ is an electron energy. Note that averaged over time third term in Eq. (7) is equal to zero due to the classical virial theorem. Thus, we conclude that in classical physics averaged over time weight of a composite body is equivalent to its energy, taken in the absence of gravitational field [8], [9].

The Hamiltonian (6) can be quantized by substituting a momentum operator, $\hat{p} = i\hbar\left(\partial/\partial r\right)$, instead of canonical momentum, $p$.

It is convenient to write the quantized Hamiltonian in the following form:

$$\hat{H} = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g \phi \quad (9)$$

where we omit term $m_e c^2$ and introduce weight operator of an electron in a weak gravitational field,

$$\hat{m}_e^g \phi = m_e \phi + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) \frac{\phi}{c^2} + \left( 2 \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right) \frac{\phi}{c^2} \quad (10)$$

Note that, in Eq. (10), the first term corresponds to the bare electron mass, $m_e$, the second term corresponds to the expected electron energy contribution to the weight operator, whereas the third non-trivial term is the virial contribution to the weight operator. It is important that the operator (10) does not commute with electron energy operator, taken in the absence of gravitational field. It is possible to show that Eqs. (9), (10) can be obtained directly from the Dirac equation in a curved spacetime, corresponding to a weak gravitational field [2].

## 3 BREAKDOWN OF THE EQUIVALENCE BETWEEN ENERGY CONTENT AND WEIGHT IN A WEAK GRAVITATIONAL FIELD FOR A QUANTUM BODY

Below, we discuss some consequences of Eqs. (9), (10). Suppose that we have a macroscopic ensemble of hydrogen atoms with each of them being in a ground state with energy $E_1$. Then, from Eq. (10), it follows that the expectation value of weight operator per atom is:

$$\left\langle \hat{m}_e^g \phi \right\rangle = m_e \phi + \frac{E_1}{c^2} \phi + \left\langle 2 \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right\rangle \frac{\phi}{c^2}$$
$$= \left( m_e + \frac{E_1}{c^2} \right) \phi \quad (11)$$

where the third term in Eq. (11) is zero in accordance with the quantum virial theorem [13]. Therefore, we conclude that the weak equivalence between weight in a weak gravitational field and energy in the absence of the field survives at a macroscopic level for stationary quantum states.

Let us discuss how Eqs. (9), (10) break the weak equivalence between weight in a weak gravitational field and energy at a microscopic level. First of all, we pay attention that the weight operator (10) does not commute with electron energy operator, taken in the absence of gravitational field. This means that, if we create a quantum state of a hydrogen atom with definite energy, it will not be characterized by definite weight. In other words, a measurement of the weight in such quantum state may give diferent values, which, as shown, are quantized.

Here, we illustrate the above mentioned inequivalence. Suppose that at $t = 0$ we create a ground state wave function of a hydrogen atom, corresponding to the absence of gravitational field:

$$\Psi_1(r,t) = \Psi_1(r) \exp\left( \frac{-iE_1 t}{\hbar} \right) \quad (12)$$

In a weak gravitational field Eq. (2), wave function Eq. (12) is not anymore a ground state of the Hamiltonian (9), (10) from point of view of an inertial observer, located at infinity. For such observer, in accordance with Eq. (3), a general solution of the Schrodinger equation, corresponding to the Hamiltonian (9), (10), can be written as:

$$\Psi(r,t) = \sum_{n=1}^{\infty} a_n \Psi_n \left[ \left( 1 - \frac{\phi}{c^2} \right) r \right] \exp\left[ -iE_n \left( 1 + \frac{\phi}{c^2} \right) \frac{t}{\hbar} \right] \quad (13)$$

[Here factor $1 - \phi/c^2$ is due to a curvature of space, whereas the term $E_n(1 + \phi/c^2)$ reflects the famous red shift in gravitational field and is due to a curvature of time. $\Psi_n(r)$ is a normalized wave function of an electron in a hydrogen atom in the absence of gravitational field, corresponding to energy $E_n$ [13].

In accordance with the quantum mechanics, probability that at $t > 0$ an electron occupies excited state with energy $E_n(1 + \phi/c^2)$ is:

$$P_n = |a_n|^2$$

where:

$$a_n = \int \Psi_1^*(r) \Psi_n \left[ \left( 1 - \frac{\phi}{c^2} \right) r \right] d^3 r$$
$$= -\left( \frac{\phi}{c^2} \right) \int \Psi_1^*(r) r \Psi_n'(r) d^3 r \quad (14)$$

for $n \neq 1$.

Taking into account that the Hamiltonian is the Hermitian operator, it is possible to show that:

$$\int \Psi_1^*(r) \Psi_n'(r) d^3 r = \frac{V_{(n,1)}}{\hbar \omega_{(n,1)}} \hbar \omega_{(n,1)}$$
$$= E_n - E_1 \quad (15)$$

where:

$$V_{(n,1)} = \int \Psi_1^*(r) \hat{V}(r) \Psi_n(r) d^3 r$$
$$\hat{V}(r) = 2 \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \quad (16)$$

Let us discuss Eqs. (13)-(16). Note that they directly

demonstrate that there is a finite probability:

$$P_n = |a_n|^2$$
$$= \left(\frac{\phi}{c^2}\right)^2 \left(\frac{V_{(n,1)}}{E_n - E_1}\right)^2 \qquad (17)$$

for $n \neq 1$, that at $t > 0$ an electron occupies n-th energy level.

In fact, this means that measurement of weight in a weak gravitational field in a quantum state with a definite energy (12) gives the following quantized values:

$$m_e^g(n)\phi = m_e\phi + \left(\frac{E_n}{c^2}\right)\phi \qquad (18)$$

corresponding to the probabilities (17) [14].

[Note that $\hat{V}(r)$ in Eq. (16) is the virial operator. It is a part of the weight operator (9), which does not commute with energy operator, taken in the absence of gravitational field. Due to the fact that $\hat{V}(r)$ presents in Eqs. (9), (16), the probabilities (17) for the quantization law (18) are not equal to zero.

We point out that, although the probabilities (15) are quadratic with respect to gravitational potential and, thus, small, the changes of the weight (18) are large and of the order of $\alpha^2 m_e$, where $\alpha$ is the fine structure constant.

We also pay attention that small values of probabilities (15), $P_n \Box 10^{-18}$, do not contradict to the existing Eotvos type measurements [11], which have confirmed the weak equivalence principle with the accuracy of the order of $10^{-12} - 10^{13}$.

For us, it is very important that the excited levels of a hydrogen atom spontaneously decay with time, therefore, one can detect quantization law (18) by measuring electromagnetic radiation, emitted by a macroscopic ensemble of hydrogen atoms [15].

The above mentioned optical method is much more sensitive than the Eotvos type measurements and we, therefore, hope that it will allow to detect the breakdown of the equivalence between energy content and weight in a weak gravitational field, suggested in the paper.

## 4 CONCLUSION

To summarize, we have demonstrated that weight of a composite quantum body in a weak external gravitational field is not equivalent to its energy in the weak sense due to quantum fluctuations.

We have also shown that the corresponding expectation values are equivalent to each other for stationary quantum states. In this context, we need to make the following comment.

First of all, we stress that, for superpositions of stationary states, the expectation values of the weight can be oscillatory functions of time even in case, where the expectation value of energy is constant.

For instance, as follows from Eq. (9), for electron wave function:

$$\Psi_{(1,2)}(r,t) = \frac{1}{\sqrt{2}}\left[\Psi_1(r)\right]\exp\left(-iE_1 t\right) + \Psi_2 \exp\left(-iE_2 t\right) \quad (19)$$

which is characterized by the time independent expectation value of energy, $\langle E \rangle = (E_1 - E_2)/2$, the expectation value of electron weight is the following oscillatory function:

$$\langle \hat{m}_e^g \phi \rangle = m_e\phi + \frac{E_1 + E_2}{2c^2}\phi + \frac{V_{(1,2)}}{c^2}\phi \cos\left[\frac{E_1 - E_2 t}{\hbar}\right] \quad (20)$$

Note that the oscillations of the weight directly demonstrate inequivalence of the weight and energy at a macroscopic level.

It is important that these oscillations are strong (of the order of $\alpha^2 m_e$ and of a pure quantum origin without classical analogs.

If we average the oscillations over time, we obtain the modified weak equivalence principle between the averaged over time expectation value of the weight and the expectation value of energy in the following form:

$$\langle\langle \hat{m}_\hat{e}^g \phi \rangle\rangle_t = m_e\phi + \frac{(E_1 + E_2)}{2c^2}\phi \qquad (21)$$

We pay attention that physical meaning of averaging procedure in Eq. (21) is completely diferent from that in classical time averaging procedure (1) and does not have the corresponding classical analog.

In conclusion, we stress that we have considered in the paper a point-like [16] composite quantum test body and all our results are due to diferent couplings of kinetic and potential energies with an external gravitational field.

This physical mechanism is completely diferent from those, considered before, where a possibility of a break-down of the weak equivalence principle was discussed due to three mass dependent phenomena: penetration of the de Broglie waves in classically restricted areas, bound states of particles in an external gravitational field, and the interference of the de Broglie waves. In addition, we point out that there exists an alternative point of view (see, for example, [17], [18], stating that there cannot be violations due to quantum efects of some generalized weak equivalence principle in any metric theory of gravitation, including the GR.

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[12]   We pay attention that to calculate the Lagrangian (and later - the Hamiltonian) in a linear with respect to a small parameter $\phi(R)/c^2$ approximation, we do not need to keep the terms of the order of $\left[\phi(R)/c^2\right]^2$ in metric (2), in contrast to the perihelion orbit procession calculations

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[14]   We pay attention that movement with constant velocity, gravitational force, which acts on each individual hydrogen atom, is compensated by some non-gravitational forces. [This causes very small changes of a hydrogen atom energy levels and is not important for our calculations.] Therefore, the atoms do not feel directly gravitational acceleration, $g$, but feel, instead, gravitational potential, $\phi(R'+vt)$, changing with time due to motion in the Earth gravitational field.

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